Module EE.PE.2.E3

System Control Overview and Economic Dispatch Calculation

Primary Authors: Gerald B. Sheble, Iowa State University
Email Address: gsheble@iastate.edu
James D. McCalley, Iowa State University
Email Address: jdm@iastate.edu
Last Update: 8/1/99
Prerequisite Competencies: Matrix Algebra, Partial Derivatives. Also, Synchronous Generator Operation, found in module EE.PE.2.G1.
Module Objectives:
1. Model the generator cost rate as a function of generator output
2. Apply the Karush-Kuhn-Tucker (KKT) conditions in solving multi-variable, constrained optimization problems.
3. Solve the economic dispatch problem by applying graphical and Newton approaches.
4. Identify the meaning of incremental cost and how it relates to Lagrange multipliers.

E3.1 Introduction

The daily operation of the electric transmission grid is primarily concerned with the balance of satisfying the demand for electricity with the supply. This is accomplished keeping in mind adherence to all rules of physics and acceptable operation for security and reliability, while simultaneously minimizing the cost of electricity production.

The main objective of this material is to describe the calculation procedures used in allocating demand among available units at minimum cost to the generation firm. In preparation for achieving this objective, we will first present the operational and functional structure in which the calculation is done. We will describe how to model generator costs. We will then study analytical procedures used in minimizing multivariate functions under constraints. At this point we will be ready to solve the economic dispatch problem.

E3.2 Operational Structure

The work of the Independent System Operator (ISO) revolves around of a facility called an Energy Control Center(ECC). Through the ECC, the ISO operates the transmission grid to provide maximum access to all members of the system within the established operational guidelines. The ISO consists of system operators, of operational planners, and of operational auditors working to track all schedules and all accounts as to the planned operation and to the actual operation. The system operators are responsible for the switching operations to isolate equipment for safety or for maintenance. The system operators are also responsible for the control of the generation to implement the contracted schedules. The operational planners establish schedules according to the contracts and bids offered by the GENCOs, TRANSCOs and DISTCOs. This provides that the transmission grid can operate within the established operational guidelines. The system operators implement the planned schedules and adjust the auxiliary services to meet the actual grid requirements. The end result is a log of all operational events to implement the schedules and all deviations from planned schedules. The operational auditors verify all schedule compliance, adjust any log entries for incomplete or missing data, compute any deviations from the planned schedules, log any needed operational changes for future plans, any deviations and the required remedies, and document all restrictions to the transmission grid transfer capability. The computer capability to support the above functions is a major reason for the research and development now occurring around the world.

There is an ISO for each location operated as a control area. The control areas are interconnected within the two major regions of the United States: East and West. Each control area is to operate as a whole, absorbing any demand changes within the control area, effectively isolating each area. The areas are interconnected to provide the capability to trade resources between areas when economically feasible and to provide resources, as system needs change until the communication and control systems can respond. The changes on the electrical grid travel nearly at the speed of light. Thus, changes must be accommodated immediately. The communication and control systems respond on the order of seconds, far too slow to respond to changes as they occur. Thus the system has to be designed to inherently, properly respond to changes as a natural response.

E3.2.1 NERC Guidelines

The security and reliability of the present electric power grid is preserved by the consensus of all electric utilities through the National Electric Reliability Council (NERC). NERC is divided into several regional groups to oversee the compliance of each company to the agreed operational requirements. NERC is responsible for the standards generation and evaluation of operating and planning standards.

Two fundamental definitions provided by NERC are as follows :

- Adequacy is the ability of the electric systems to supply the aggregate electrical demand and energy requirements of customers at all times taking into account scheduled and reasonably expected unscheduled outage of system elements.
- Security is the ability of the electric system to withstand sudden disturbances such as electric short circuits and unanticipated loss of system elements.

E3.3 Energy Management System Overview

The energy control center (ECC) is a facility where the system operators can analyze and operate the transmission grid through a set of software applications called an energy management system (EMS). There are two basic EMS functions; security monitoring and control / dispatch.

In security monitoring, the state of the power system is classified into one of the following: secure, alert, alarm, compromised. A secure state is when the system is operating as planned with no immediate probable problems. An alert state is when the system is operating as planned with immediate problems probable. An alarmed state is when the power system is operating outside acceptable levels. A compromised state is when the power system is operating outside allowable levels and outside acceptable schedule deviations. Network analysis functions operate periodically to determine the state of the power system (e.g. every five minutes) for the planned schedules and operating conditions. Alternatively, the operator may analyze the power system under a hypothetical situation to determine the state of the power system on a demand (or as needed) basis.

The control and dispatch of the generation is directed through the energy management system. The functions which implement this direction are the Automatic Generation Control (AGC) and the Economic Dispatch calculations (EDC). These functions determine the set points of the governors and allocate the demand among generators. AGC operates continuously as a feedback control system that senses instantaneous frequency deviations caused by power imbalance and adjusts generator MW output to compensate the power imbalance. This is accomplished by the action of the governor at each generator (see module G1). EDC operates such more slowly, sensing steady-state frequency and tie-line flow deviation every 3-5 minutes and readjusting all generator MW outputs accordingly. We will focus on the EDC approach in this module. A fundamental part of this approach is the cost of generators electrical energy.

E3.4 Costs of Generating Electrical Energy

The costs of electrical energy generation arise mainly from three sources: facility construction, ownership costs, and operating costs. The last is the most significant portion of power system operation, and in this section we focus on this aspect.

E3.4.1 Operating Costs

These costs include the costs of labor, but they are dominated by the fuel costs necessary to produce electrical energy (MW) from the plant. Some typical average costs of fuel, as of 1989, are given in the following table for the most common types in use today.

Table E3.1

Fuel Type	$/MBTU
Coal	1.50
Oil	3.00
Uranium	0.65
Natural Gas	2.35

These values do not reflect the actual costs of producing electrical energy because substantial losses occur during production. Some power plants have overall efficiencies as low as 35%; in addition, the plant efficiency varies as a function of the generation level P_g. We illustrate this point in what follows.

We represent plant efficiency by η. Then η=energy output/energy input. We can actually obtain η as a function of P_g by measuring the energy output of the plant in MWHRS and the energy input to the plant in MBTU. For example, we could get the energy output by using a wattmeter to obtain P_g as a function of time and then compute the area under the curve for an interval, and we could get the energy input by measuring the coal tonnage used during the interval and then multiply by the coal energy content in MBTU/ton). Then η is proportional to the ratio of MWHR/MBTU; a plot of this ratio versus P_g would appear as in Figure E3.1.

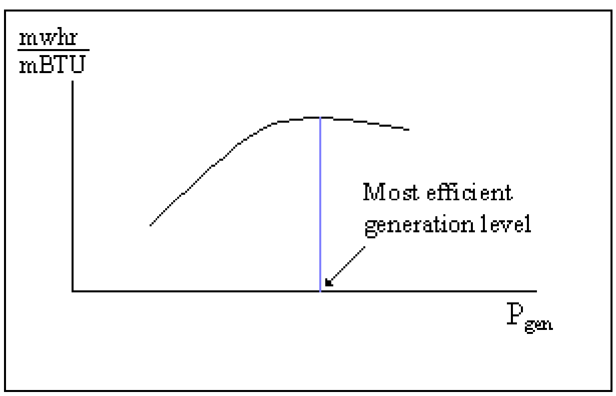

Figure E3.1 Plot of MWhr/MBTU (proportional to efficiency) vs. Generation (Pg)

Figure E3.1 indicates that efficiency is poor for low generation levels and increases with generation, but at some optimum level it begins to diminish. Most power plants are designed so that the optimum level is at or close to the rated output.

The *heat rate* curve is similar to Figure E3.1 except that the y-axis is inverted to yield MBTU/MWhrs, which is proportional to $1/\eta$. This curve is illustrated in Figure E3.2. Heat rate is denoted by H. Since the heat rate is dependent on operating point, we write $H = H(P_g)$. Some typical heat rates for units at maximum output are (in MBTU/MWhrs) 9.5 for fossil-steam units, 10.5 for nuclear units, and 13.0 for combustion turbines [1].

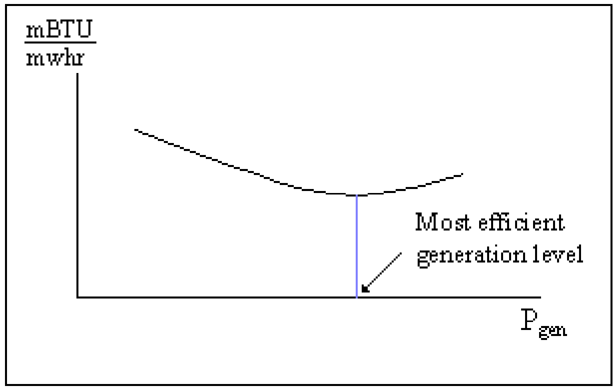

Figure E3.2 Plot of Heat Rate (H) vs. Generation (Pg)

We are primarily interested in how the cost per MWHR changes with P_g. We assume that we know K, the cost of the input fuel in \$/MBTU. Define R as the rate at which the plant uses fuel, in MBTU/hr (which is dependent on P_g), and C as the cost per hour in \$/hour. Then R = $P_g H(P_g)$ and C = (R)(K) = $P_g H(P_g)K$. A plot of C vs. P_g is illustrated in Figure E3.3.

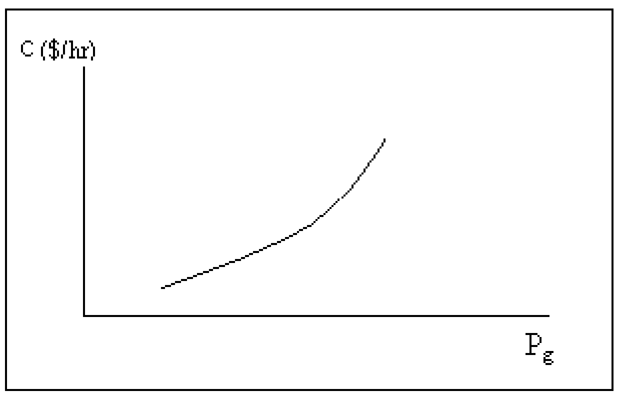

Figure E3.3 Plot of Cost per Hour (C) vs. Generation (Pg)

The desired \$/MWHR characteristic, called the incremental cost curve for the plant, can be obtained by differentiating the plot in Figure E3.3. The incremental cost curve is shown in Figure E3.4.

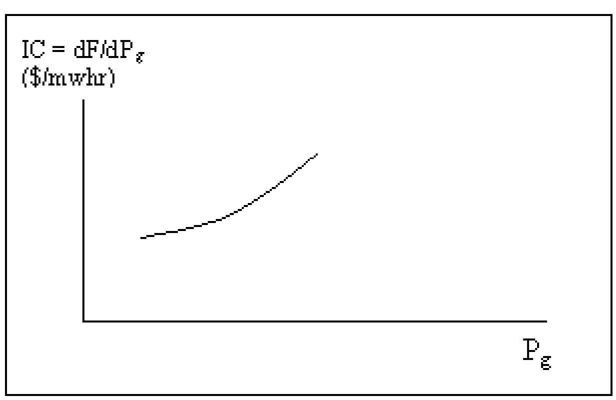

Figure E3.4 Plot of Incremental Cost (IC) vs. Generation (Pg)

Example E 3.1

A 100 MW coal-fired plant uses a type of coal having an energy content of 12,000 BTU/lb (the conversion factor from joules to BTU is 1054.85 joules/BTU). The coal cost is $1.5/MBTU. Typical coal usage corresponding to the daily loading schedule for the plant is as follows:

Table E3.2

Time of Day	Electric Output (MW)	Coal Used (tons)
12:00am-6:00am	40	105.0
6:00am-10:00am	70	94.5
10:00am-4:00pm	80	156.0
4:00pm-12:00am	100	270.0

For each of the four load levels, find (a) the efficiency, η (b) the heat rate H (MBTU/MWhr) (c) the cost per hour, C ($/hr). Also, for the loading levels of 40, 70, and 80 MW, use a piecewise linear plot of F vs P to obtain incremental costs.

Solution

Let T be the number of hours the plant is producing P MW while using y tons of coal.

(a) $$\eta = \frac{P \times T \times 3600 \frac{\sec}{hr} \times 10^6 \frac{watts}{MW}}{12{,}000 \frac{BTU}{lb} \times 2000 \frac{lb}{ton} \times 1054.85 \frac{joules}{BTU} \times ytons}$$

Note that the above expression for efficiency is dimensionless.

(b) $$H = \frac{12{,}000 \frac{BTU}{lb} \times 2000 \frac{lb}{ton} \times ytons \times \frac{1MBTU}{10^6 BTU}}{P \times T}$$

Note that $H = \frac{1}{\eta} \times \frac{3600}{1054.85} = \frac{3.41}{\eta}$, and the above expression has units of MBTU/MWhr.

(c) C = (R)K where R is the rate at which the plant uses fuel and K is fuel cost in $/MBTU. Note from units of *P* and *H* that R = (P)(H) ➔ F = (P)(H)(K) where *H* is a function of *P*.

Application of these expressions for each load level yields the following results:

Table E3.3

T (hrs)	P (MW)	y (tons)	Efficiency	H (mbtu/mwhr)	C ($/hr)
6	40	105.0	0.33	10.5	630
4	70	94.5	0.42	8.1	850
6	80	156.0	0.44	7.8	936
8	100	270.0	0.42	8.1	1215

To obtain incremental cost $IC = \frac{dC}{dP}$, we plot C vs. *P* and then get an approximation on the derivative by assuming a piecewise linear model as shown in Figure E3.5.

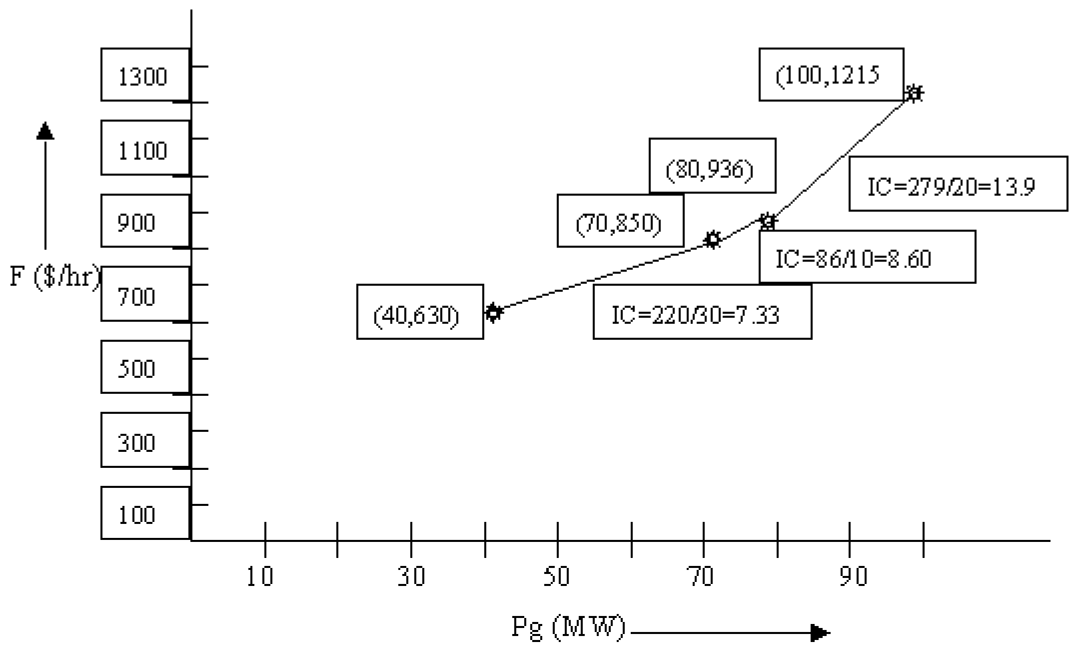

Figure E3.5 Calculation of Incremental Cost

E3.4.2 Facility Construction Costs and Ownership Costs

Construction costs include the costs of all necessary labor and materials necessary to plan, gain regulatory approval, and construct new generation facilities. In the past, utilities were able to minimize these costs by building fewer, but larger facilities, due to economies of scale. This is no longer the case for the following reasons:

- Smaller plants can be built more quickly and their construction costs are consequently subject to less uncertainty.
- Smaller plants can be located closer to load centers. This attribute decreases system losses and tends to be advantageous for system security.
- Cogeneration facilities are attractive because of their high efficiency. They typically have lower ratings as a result of their dependency on the industrial steam processes supporting or supported by them.
- Plants fueled by renewable energy sources (biomass, wind, solar, and independent hydro) are attractive because of their low operating costs and environmental appeal. They also tend to have lower ratings.

Ownership costs are not related to how much the plant is used. They arise simply because the plant exists, and they include maintenance and capital costs. Capital costs include insurance, depreciation, taxes, and administrative expenses. These costs are sometimes called “existing facilities costs” or “embedded costs.”

E3.5 Optimization Overview with Economic Dispatch Examples

Optimization problems occur in many different fields. There is, in fact, one field, namely operations research, which is dedicated entirely to the study of posing and solving optimization problems. Perhaps the most common application is to identify the least expensive way of satisfying a demand. The airline, telephone, and manufacturing industries are good examples of industries that make heavy use of optimization. Another good example is, of course, the electric power industry.

Module G1.1 defined a model for representing the operating costs of generation. Here we approximate the cost rate vs. generation (Figure E3.5) curve using a quadratic function.

E3.5.1 Introduction

Economic dispatch is the process of allocating the required load demand between the available generation units such that the cost of operation is at a minimum. One-dimensional minimization problems are covered in a basic calculus course. The Economic Dispatch problem is a more general type of optimization problem. We will see that the Economic Dispatch problem is a non-linear, multivariable, constrained optimization problem.

Nonlinear optimization techniques can be divided by type: unconstrained search, linearly constrained search, quadratic objective programming, convex programming, separable convex programming, nonconvex programming, geometric programming, fractional programming, etc. It is easier to classify the techniques by the type of problem to be solved:

a. Linear objective function, linear constraints
b. Nonlinear objective function, linear constraints
c. Nonlinear objective function, nonlinear constraints
d. Linear objective function, nonlinear constraints

The type “a” problem is most often solved with Linear Programming techniques based on the Simplex method. Approximating the nonlinear objective function often solves the type “b” problems. If the nonlinear objective function is of a definite form, then a specialized technique may be used. If the objective function is a quadratic function, then Quadratic Programming is appropriate. If the objective function is piece-wise linear, then a separable function is appropriate. Functional characteristics, such as convexity, may simplify the technique and correspondingly accelerate convergence to the optimal solution. The Reduced Gradient method is best for general problems of this type.

The type "c" problems are the hardest to solve. Typically, unless the functions demonstrate simplifying characteristics, the nonlinear functions are approximated or an alternative sequence of approximating problems is solved. When an alternative sequence of approximate problems is solved, it is assumed that the final approximate problem replicates the original problem. The General Reduced Gradient method is best for general problems of this form. The type "d" problems are almost as hard to solve as the type "c" problems since there is only one objective function and many constraints. The non-linearity of the many overshadows the linearity of the one. The Convex Simplex (LP) method is best for general problems of this form.

The solution methods presented in this text are the analytical method and the graphical method (also known as the LaGrangian Relaxation method). The analytical technique solves the optimality conditions as a set of simultaneous equations to find the solution. The graphical technique uses the conditions of optimality for estimating where the solution should be, moves to that solution point and then re-estimates where the solution should be. If the solution point is where it was predicted, then the process has found the optimal solution.

The following sections present the basic principles upon which these solution techniques depend. The Economic Dispatch problem will be used to illustrate the similarities and the differences between the techniques.

Note that underlined letters are used to denote vectors ($\underline{x}$) or arrays ($\underline{A}$). The n decision variables $\underline{x}$ will use the subscript “i”; the m equality constraints will use the subscript “j”; the r inequality constrains will use the subscript “k”. The objective function will be represented by f, and the constraint(s) by $\underline{h}$ (equality) and $\underline{g}$ (inequality).

Decision variables are the parameters that can be changed through control and communication systems. All other variables are dependent on decision variables. The relationship between the decision variables and the dependent variables are found in the constraints. The objective function describes the improvement as a function of the decision and dependent variables. Inequality constraints typically represent the limitations of equipment (e.g., maximum capacity). Equality constraints normally represent physical laws (e.g., conservation of energy).

E3.5.2 General Optimization Problem Statement

The general form of a nonlinear programming problem is to find $\underline{x}$ so as to:

$$\text{Min } f(\underline{x}) \qquad \text{(E3.1)}$$

$$\text{subject to: } \quad \underline{g}(\underline{x}) \leq \underline{b}$$
$$\underline{h}(\underline{x}) = \underline{c}$$
$$\text{and: } \quad \underline{x} \geq 0$$

where f, $\underline{g}$, and $\underline{h}$ are given functions of the *n* decision variables $\underline{x}$. Note that the condition $\underline{x} \geq \underline{0}$ can be satisfied by appropriate definition of decision variables.

This text does not attempt to survey the general optimization problem. This is a large research area with many texts appropriate for further study. Considerable research is continuing in this area and will continue for some time.

There are no absolutes in the area of Nonlinear Optimization. Previous and new techniques can only be assessed by trial and error. Previously judicated good techniques may no longer be appropriate as new constraints or parameter changes are needed. Fortunately, at least the necessary conditions for an optimum to exist can be identified, most of the time.

E3.5.3 KKT Conditions and LaGrangian Multipliers

The first step is to form the LaGrangian function of (E3.1):

$$F(\underline{x},\underline{\lambda},\underline{\mu}) \equiv f(\underline{x}) - \underline{\lambda}^T[\underline{h}(\underline{x}) - \underline{c}] - \underline{\mu}^T[\underline{g}(\underline{x}) - \underline{b}] \qquad \text{(E3.2)}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2 \ldots, \lambda_m)$ and $\underline{\mu} = (\mu_1, \mu_2 \ldots, \mu_r)$ are called dual variables. The LaGrangian function is simply the summation of the objective function with the constraints. It is assumed that f, $\underline{h}$, and $\underline{g}$ are continuous and differentiable. Given that $\underline{x}$ is a feasible point, the conditions for which the optimal solution occurs are:

$$\frac{\partial F}{\partial x_i} = 0 \qquad \forall\ i = 1, n \qquad \text{(E3.3a)}$$

$$\frac{\partial F}{\partial \lambda_j} = 0 \qquad \forall\ j = 1, m \qquad \text{(E3.3b)}$$

$$\mu_k [g_k(x) - b] = 0 \qquad \forall\ k = 1, r \qquad \text{(E3.3c)}$$

$$x_i \geq 0 \qquad \forall\ i = 1, n \qquad \text{(E3.3d)}$$

$$\mu_k \geq 0 \qquad \forall\ k = 1, r \qquad \text{(E3.3e)}$$

These conditions are known as the Karush-Kuhn-Tucker (KKT) conditions or, more simply, as the Kuhn-Tucker (KT) conditions. The KKT conditions state that for an optimal point

1) The derivatives of the LaGrangian with respect to all decision variables must be zero (Eq. E3.3a).
2) All equality constraints must be satisfied (Eq. E3.3b).
3) A multiplier μ_k cannot be zero when its corresponding constraint is binding (Eq E3.3c).
4) All decision variables must be non-negative at the optimum (Eq.E3.3d).
5) All multipliers μ_k must be non-negative (Eq. E3.3e).

Requirement 3, corresponding to E3.3c, is called the "complementary" condition. The complementary condition is very important to understand. If $\underline{x}$ occurs on the boundary of the k^{th} inequality constraint, then $g_k(\underline{x}) = b_k$. In this case

$$g_k(\underline{x}) \rightarrow h_{m+1}(\underline{x})$$

$$\mu_k \rightarrow \lambda_{m+1}$$

Eqn.(E3.3c) allows μ_k to be non-zero. Once it is known that the k^{th} constraint is binding, then the k^{th} constraint can be moved to the vector of equality constraints; i.e. $g_k(\underline{x})$ can then be renamed as $h_{m+1}(\underline{x})$ and μ_k as λ_{m+1}.

On the other hand, if the solution x does not occur on the boundary of the k^{th} inequality constraint, then (assuming $\underline{x}$ is an attainable point) $g_k(\underline{x}) - b_k < 0$. In this case, Eq. E3.19c requires that $\mu_k = 0$ and the k^{th} constraint makes no contribution to the LaGrangian.

It is important to understand the significance of μ and λ. The optimal values of the LaGrangian Multipliers are in fact the rates of change of the optimum attainable objective value $f(\underline{x})$ with respect to changes in the right-hand-side elements of the constraints. Economists know these variables as shadow prices or marginal values. This information can be used not only to investigate changes to the original problem but also to accelerate repeat solutions. The marginal values λ_j or μ_k indicate how much the objective $f(\underline{x})$ would improve if a constraint b_j or c_k, respectively, were changed. One constraint often investigated for change is the maximum production of a plant.

This is the limit of optimization theory to be presented. The interested reader is referenced to one of the texts in the references [1-8].

E3.6 Economic Dispatch Formulation

Economic Dispatch is the process of allocating the required load demand between the available generation units such that the cost of operation is minimized. There have been many algorithms proposed for economic dispatch: Merit Order Loading, Range Elimination, Binary Section, Secant Section, Graphical/Table Look-Up, Convex Simplex, Dantzig-Wolf Decomposition, Separable Convex Linear Programming, Reduced Gradient with Linear Constraints, Steepest Descent Gradient, First Order Gradient, Merit Order Reduced Gradient, etc. The close similarity of the above techniques can be shown if the solution steps are compared. These algorithms are well documented in the literature. We will use only the analytical and the graphical (LaGrangian Relaxation) techniques.

Economic Dispatch is also the most intensive part of a Unit Commitment program. An Economic Dispatch algorithm expends approximately seventy (70) percent of the computer time of a Unit Commitment program [1]. Thus, the selection and implementation of an Economic Dispatch algorithm is a central issue of any Unit Commitment research. Also, since Economic Dispatch executes approximately once every five minutes in each energy control center, any computation reduction has a significant impact. Thus, it is necessary for the selection of the best method for Economic Dispatch.

This text is directed to introduce the optimization algorithms in the general literature. Thus, the following does not address the selection of the best method to use for Economic Dispatch for a given problem or data. However, the following does provide an excellent starting point.

E3.6.1 Generation Models

The electric power system representation for Economic Dispatch consists of models for the generating units and can also include models for the transmission system. The generation model represents the cost of producing electricity as a function of power generated and the generation capability of each unit. This model was discussed in section 3.4 of this module. We can specify it as:

1. Unit cost function:

$$COST_i = C_i(P_i) \qquad \text{(E3.4)}$$

where $COST_i$ = production cost (units of $/hr)

C_i = energy to cost conversion curve

P_i = production power

2. Unit capacity limits:

$$P_i \geq \underline{P}_i$$
$$P_i \leq \overline{P}_i \qquad \text{(E3.5)}$$

where

$$\underline{P}_i = P_{\min} = \min \ \textit{generator level}$$
$$\overline{P}_i = P_{\max} = \max \ \textit{generator level}$$

E3.6.2 Transmission Model

The general transmission model used for EDC represents the balance between power supplied and power consumed within and delivered from the area of the interconnection for which the calculation is being done. This area of connection is hereafter referred to as the “control area”. In general, then, we may write

where P_i is the power generation at unit “i”, P_D is the total power demanded in the control area, P_{LOSS} is the total power loss in the control area, and P_{tie} is the total power flowing out of the control area into other interconnected control areas. If the power is flowing in, then P_{tie} is a negative number.

$$\sum_{i=1}^{n} P_i = P_D + P_{Loss} + P_{tie} \qquad \text{(E3.6)}$$

Let’s assume that for a given demand P_D and tie flow P_{tie} the losses are fixed. This is an approximation because in reality the losses will change depending on how power demand is allocated to the various generators. We accept this approximation here in order to keep the discussion basic. We note that Eqn.(E3.6) represents an equality constraint. It is sometimes called the power balance constraint.

[1] Unit commitment is the procedures used to determine which generation units should be connected to the grid.

E3.6.3 Formulation of the LaGrangian

We are now in a position to formulate our optimization problem. Stated in words, we desire to minimize the total cost of generation subject to the inequality constraints on individual units (E3.5) and the power balance constraint (E3.6). Stated analytically, we have:

Minimize: $\sum_{i=1}^{n} C_i(P_i)$

Subject to:
$$\sum_{i=1}^{n} P_i = P_D + P_{Loss} + P_{tie} = P_T$$
$$P_i \geq \underline{\underline{P_i}} \Rightarrow -P_i \leq -\underline{\underline{P_i}}$$
$$P_i \leq \overline{\overline{P_i}} \qquad \text{(E3.7)}$$
$$P_i \geq 0$$

We note that this optimization problem is in the same form as E3.1 if we recognize these similarities:

$$P_i \rightarrow x_i$$
$$\sum_{i=1} C_i(P_i) \rightarrow f(\underline{x})$$
$$P_T = P_D + P_{LOSS} + P_{tie} \rightarrow \underline{c}$$
$$\sum_{i=1}^{n} P_i \rightarrow h$$
$$-P_i \leq -\underline{\underline{P_i}}, \quad P_i \leq \overline{\overline{P_i}} \quad \forall\, i = 1, n \rightarrow \underline{g}(\underline{x}) \leq \underline{b}$$

The equality constraint $\underline{h}(\underline{x}) = \underline{c}$ for the general case was allowed to contain multiple constraints. Here, in the EDC problem, we see that there is only one equality constraint, i.e., $\underline{h}$ and $\underline{c}$ are both scalars. This implies that λ is a scalar also. The LaGrangian function, then, is:

$$\begin{aligned} F(\underline{P}, \lambda, \underline{\mu}) = & \sum_{i=1}^{n} C_i(P_i) - \lambda\left[\sum_{i=1}^{n} P_i - P_T\right] \\ & - \underline{\underline{\mu}}_1\left[-P_1 + \underline{\underline{P}}_1\right] - \overline{\overline{\mu}}_1\left[P_1 - \overline{\overline{P}}_1\right] \\ & - \underline{\underline{\mu}}_2\left[-P_2 + \underline{\underline{P}}_2\right] - \overline{\overline{\mu}}_2\left[P_2 - \overline{\overline{P}}_2\right] \qquad \text{(E 3.8)} \\ & - \bullet\bullet\bullet\bullet \\ & - \underline{\underline{\mu}}_n\left[-P_n + \underline{\underline{P}}_n\right] - \overline{\overline{\mu}}_n\left[P_n - \overline{\overline{P}}_n\right] \end{aligned}$$

Here, we note that r = 2n (n is the number of generators) because there are 2 inequality constraints for each decision variable P_i: the maximum and minimum levels of generation.

E3.6.4 KKT Conditions

Application of the KKT conditions to the LaGrangian function of E.3.2 results in:

$$\frac{\partial F}{\partial P_i} = 0 \quad \Rightarrow \quad \frac{\partial C_i(P_i)}{\partial P_i} - \lambda + \underline{\underline{\mu}}_i - \overline{\overline{\mu}}_i = 0 \quad \forall i = 1, n \tag{E3.9}$$

$$\frac{\partial F}{\partial P_i} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} P_i - P_D - P_{LOSS} - P_{tie} = 0 \tag{E3.10}$$

$$\begin{aligned} \underline{\mu}^{T}\left[\underline{g}(\underline{x}) - \underline{b}\right] = \underline{0} \quad \Rightarrow \quad & \underline{\underline{\mu}}_1\left[-P_1 + \underline{\underline{P}}_1\right] = 0,\ \overline{\overline{\mu}}_1\left[P_1 - \overline{\overline{P}}_1\right] = 0 \\ & \underline{\underline{\mu}}_2\left[-P_2 + \underline{\underline{P}}_2\right] = 0,\ \overline{\overline{\mu}}_2\left[P_2 - \overline{\overline{P}}_2\right] = 0 \\ & .... \\ & \underline{\underline{\mu}}_n\left[-P_n + \underline{\underline{P}}_n\right] = 0,\ \overline{\overline{\mu}}_n\left[P_n - \overline{\overline{P}}_n\right] = 0 \end{aligned} \tag{E3.11}$$

$$\mu_i \geq 0 \quad \forall\ i = 1, n \quad \Rightarrow \quad \underline{\underline{\mu}}_i \geq 0,\ \overline{\overline{\mu}}_i \geq 0 \quad \forall\ i = 1, n$$

The KKT conditions provide us with a set of equations that can be solved. The unknowns in these equations include the generation levels P_1, P_2, …, P_n and the LaGrange multipliers,

$$\lambda,\ \underline{\underline{\mu}}_1,\ \underline{\underline{\mu}}_2,\ \ldots\ \underline{\underline{\mu}}_n,\ \overline{\overline{\mu}}_1,\ \overline{\overline{\mu}}_2,\ \ldots\ \overline{\overline{\mu}}_n$$

a total of (3n+1) unknowns. We note that E3.9 provides n equations, E3.10 provides one equation, and E3.11 provides (2n) equations. Thus, we have a total of (3n+1) equations.

E3.6.5 KKT Conditions for a 2-Unit System

To illustrate more concretely, let's consider a simple system having only two generating units. The LaGrangian function, from E3.8, is:

$$\begin{aligned} F\left(P_1, P_2, \lambda, \underline{\underline{\mu}}_1, \overline{\overline{\mu}}_1, \underline{\underline{\mu}}_2, \overline{\overline{\mu}}_2\right) \quad = \quad & C_1(P_1) + C_2(P_2) \\ & - \lambda\left[P_1 + P_2 - P_T\right] \\ & - \underline{\underline{\mu}}_1\left[-P_1 + \underline{\underline{P}}_1\right] - \overline{\overline{\mu}}_1\left[P_1 - \overline{\overline{P}}_1\right] \\ & - \underline{\underline{\mu}}_2\left[-P_2 + \underline{\underline{P}}_2\right] - \overline{\overline{\mu}}_2\left[P_2 - \overline{\overline{P}}_2\right] \end{aligned} \tag{E3.12}$$

The KKT conditions, from E3.9, E3.10, and E3.11 become:

$$\frac{\partial F}{\partial P_1} = 0 \quad \Rightarrow \quad \frac{\partial C_1(P_1)}{\partial P_1} - \lambda + \underline{\underline{\mu}}_1 - \overline{\overline{\mu}}_1 = 0 \tag{E3.13}$$

$$\frac{\partial F}{\partial P_2} = 0 \quad \Rightarrow \quad \frac{\partial C_2(P_2)}{\partial P_2} - \lambda + \underline{\underline{\mu}}_2 - \overline{\overline{\mu}}_2 = 0 \tag{E3.14}$$

$$\frac{\partial F}{\partial \lambda} = 0 \quad \Rightarrow \quad P_1 + P_2 - P_T = 0 \qquad \text{(E3.15)}$$

$$\underline{\mu}^{\mathrm{T}}\left[\underline{h}(\underline{x}) - \underline{c}\right] = \underline{0} \quad \Rightarrow \quad \underline{\underline{\mu}}_1\left[-P_1 + \underline{\underline{P}}_1\right] = 0,\ \overline{\overline{\mu}}_1\left[P_1 - \overline{\overline{P}}_1\right] = 0$$

$$\underline{\underline{\mu}}_2\left[-P_2 + \underline{\underline{P}}_2\right] = 0,\ \overline{\overline{\mu}}_2\left[P_2 - \overline{\overline{P}}_2\right] = 0$$

We see that there are seven unknowns:

$$P_1, P_2, \lambda, \underline{\underline{\mu}}_1, \underline{\underline{\mu}}_2, \overline{\overline{\mu}}_1, \overline{\overline{\mu}}_2$$

There are also seven equations.

Example E 3.2

Let's now provide numerical data for the two-unit problem. The 'cost-curves' are approximated using quadratic functions. In general, the form of these functions is given by:

$$C_i(P_i) = a_i(P_i)^2 + b_i P_i + c_i \qquad \text{(E3.16)}$$

where a_i is the 'quadratic' term, b_i is the 'linear' term, and c_i is the 'constant' term. These terms, together with the minimum and maximum generation specifications for each generator are given in the table below. The total generation to be allocated is $P_T = P_D + P_{LOSS} + P_{TIE=400MW}$.

Table E3.4 Dispatch Data for Example Case

	Unit 1	Unit 2
Generation Specifications:		
Minimum Generation	200 MW	100 MW
Maximum Generation	380 MW	200 MW
Cost Curve Coefficients:		
Quadratic Term	0.016	0.019
Linear Term	2.187	2.407
Constant Term	120.312	74.074

The LaGrangian function is:

$$\begin{aligned} F\left(P_1, P_2, \lambda, \underline{\underline{\mu}}_1, \overline{\overline{\mu}}_1, \underline{\underline{\mu}}_2, \overline{\overline{\mu}}_2\right) = \; & 0.016(P_1)^2 + 2.187(P_1) + 120.312 + 0.019(P_2)^2 + 2.407(P_2) + 74.074 \\ & - \lambda\left[P_1 + P_2 - 400\right] \\ & - \underline{\underline{\mu}}_1\left[-P_1 + 200\right] - \overline{\overline{\mu}}_1\left[P_1 - 380\right] \\ & - \underline{\underline{\mu}}_2\left[-P_2 + 100\right] - \overline{\overline{\mu}}_2\left[P_2 - 200\right] \end{aligned} \qquad \text{(E3.17)}$$

The KKT conditions are then given by:

$$\frac{\partial F}{\partial P_1}=0 \quad \Rightarrow \quad 0.032(P_1)+2.187-\lambda+\underline{\underline{\mu}}_1-\overline{\overline{\mu}}_1=0 \tag{E3.18}$$

$$\frac{\partial F}{\partial P_2}=0 \quad \Rightarrow \quad 0.038(P_2)+2.407-\lambda+\underline{\underline{\mu}}_2-\overline{\overline{\mu}}_2=0 \tag{E3.19}$$

$$\frac{\partial F}{\partial \lambda}=0 \quad \Rightarrow \quad P_1+P_2-400=0 \tag{E3.20}$$

$$\underline{\mu}^{T}\left[\underline{g}(\underline{x})-\underline{c}\right]=\underline{0} \quad \Rightarrow \quad \underline{\underline{\mu}}_1[-P_1+200]=0,\ \overline{\overline{\mu}}_1[P_1-380]=0 \tag{E3.21}$$

$$\underline{\underline{\mu}}_2[-P_2+100]=0,\ \overline{\overline{\mu}}_2[P_2-200]=0 \tag{E3.22}$$

E3.7 Solution Procedures

We will study two solution procedures. The first one is analytical and the second one is graphical. We will illustrate both solution procedures by extending Example E3.2.

E3.7.1 Analytical Solution

One notes that Eqn. (E3.18), (E3.19), and (E3.20) are linear in the unknowns. However, Eqn. (E3.21) and (E3.22) are not linear due to the product terms consisting of the LaGrange multipliers and the P_i variables. In general, solving linear equations is "easy", while solving non-linear equations is not. We desire a solution approach where we can apply the mathematics of linear equations.

Recall from that Eqs. (E3.21) and (E3.22) are derived from the complementary condition. This condition requires that, in (E3.21),

$$\overline{\overline{\mu}}_1=0 \quad or \quad P_1-380=0$$

and in (E3.22),

$$\underline{\underline{\mu}}_2=0 \quad or \quad -P_2+100=0$$

and

$$\overline{\overline{\mu}}_2=0 \quad or \quad P_2-200=0$$

Our solution procedure is based on the following idea:

> For each equation associated with the complementary conditions, we can guess which term is zero. We then solve the resulting set of equations (E3.18, E3.19, and E3.20), and check to see if the solution satisfies the original inequality constraints. If it does, our guess was correct. If it does not, we make another guess and try again.

The most natural starting guess is that all inequality constraints are 'non-binding', meaning the solution has all generation levels within (but not at) their associated limits. The implication of this is that

$$\underline{\underline{\mu}}_i=0 \quad and \quad \overline{\overline{\mu}}_i=0 \qquad \forall_i=1,n$$

Example E 3.3

For the two-unit system supplying 400 MW, the KKT conditions reduce to

$$\begin{aligned} 0.032(P_1) + 2.187 - \lambda &= 0 \\ 0.038(P_2) + 2.407 - \lambda &= 0 \\ P_1 + P_2 - 400 &= 0 \end{aligned} \tag{E3.23}$$

We can rewrite these equations as

$$\begin{aligned} 0.032(P_1) + 0(P_2) - \lambda &= -2.187 \\ 0(P_1) + 0.038(P_2) - \lambda &= -2.407 \\ P_1 + P_2 &= 400 \end{aligned} \tag{E3.24}$$

In matrix form this set of equations is represented by

$$\begin{bmatrix} 0.032 & 0 & -1 \\ 0 & 0.038 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 400 \end{bmatrix} \tag{E3.25}$$

Solving this matrix equation (using MATLAB) we have

$$\begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 220.29 \\ 179.71 \\ 9.24 \end{bmatrix}$$

where P_1 = 220.29 MW, P_2 = 179.71 MW, and λ = 9.24 $/MWhr. From table E3.4 we can see that

$$\begin{aligned} 200 \le P_1 \le 350 \\ 100 \le P_2 \le 200 \end{aligned}$$

We have guessed correctly and found the solution.

Example E 3.4

It is important at this point to understand the meaning of λ = 9.24 $/MWhr. This is the system incremental cost. It indicates how the total system costs would change if we increased the demand by 1 MW for the next hour. We can check this interpretation by computing the total system cost at 400 MW and again at 401 MW. At 400 MW we have P_1 = 220.29 and P_2 = 179.71. Therefore,

$$\begin{aligned} C_1(P_1) &= 0.016(220.29)^2 + 2.187(220.29) + 120.312 \\ C_1(P_1) &= 1378.53\ \$/hr \\ C_2(P_2) &= 0.019(179.71)^2 + 2.407(179.71) + 74.074 \\ C_2(P_2) &= 1120.25\ \$/hr \end{aligned}$$

Total costs are

$$C_T = C_1(P_1) + C_2(P_2) = 1378.53 + 1120.25 = 2498.78$$

Now we need to obtain total costs for P_T = 401 MW. Again, guessing that the constraints are non-binding (guessing that our optimized solution will be within the bounds of operation for the generator),

$$\underline{\underline{\mu}}_1 = \overline{\overline{\mu}}_1 = \underline{\underline{\mu}}_2 = \overline{\overline{\mu}}_2 = 0$$

the KKT conditions reduce to

$$\begin{bmatrix} 0.032 & 0 & -1 \\ 0 & 0.038 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 401 \end{bmatrix} \qquad \text{(E3.26)}$$

The solution is

$$\begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 220.83 \\ 180.17 \\ 9.25 \end{bmatrix}$$

The costs for each generator are

$$C_1(P_1) = 0.016(220.83)^2 + 2.187(220.83) + 120.312$$
$$C_1(P_1) = 1383.52\ \$/hr$$
$$C_2(P_2) = 0.019(180.17)^2 + 2.407(180.17) + 74.074$$
$$C_2(P_2) = 1124.51\ \$/hr$$

The total cost is

$$C_T = C_1(P_1) + C_2(P_2) = 1383.52 + 1124.51 = 2508.03$$

So, as a result of the 1 MW increase in demand, the total cost will change by 2508.03-2498.78 = 9.25 \$/hr. This is in agreement with our solution of λ = 9.25 \$/hr.

Example E 3.5 (Binding Constraint)

Now let's investigate what happens when our initial guess is incorrect. Consider a total demand of P_T = 550 MW. Guessing that all constraints are non-binding, the KKT conditions reduce to

$$\begin{bmatrix} 0.032 & 0 & -1 \\ 0 & 0.038 & -1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 550 \end{bmatrix} \qquad \text{(E3.27)}$$

with a solution of

$$\begin{bmatrix} P_1 \\ P_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 301.71 \\ 248.29 \\ 11.84 \end{bmatrix}$$

Because generator 2 must generate between 100 MW and 200 MW, we see that P_2 is out of range.

So we must guess again. However, our next guess should not be made arbitrarily. The fact that P_2 is above its generation limit with constraints ignored suggests that it is a less expensive unit. Accordingly, we should try to extract as much power from it as possible. So let us set P_2 = 200 MW. Because this example includes only two units the solution may be found quite easily:

$$P_1 + P_2 - 550 = 0$$
$$P_2 = 200\ MW$$
$$\Rightarrow P_1 = 350\ M$$

However, this direct-substitution approach would not work for systems having more than 2 units since assigning a constant value to one of the three or more variables would still leave two or more variables for which to solve. In addition, it does not identify the values of the LaGrange multipliers. We will therefore proceed with the formal solution approach.

Setting P_2 = 200 MW implies that the constraint associated with

$$\bar{\bar{\mu}}_2[P_2 - 200] = 0$$

is binding. This means that μ_2-double-upper-bar may not be zero. The KKT conditions are, therefore, from E3.18, E3.19, E3.20, and E3.22.

$$\begin{aligned} &0.032(P_1) + 2.187 - \lambda = 0 \\ &0.038(P_2) + 2.407 - \lambda - \bar{\bar{\mu}}_2 = 0 \\ &P_1 + P_2 - 550 = 0 \end{aligned} \quad \text{(E3.28)}$$

Because we have three equations and four unknowns we need another equation. We know that P_2 - 200 = 0. This provides our fourth equation. Rewriting all four equations we have

$$\begin{aligned} &0.032(P_1) + 0(P2) - \lambda = -2.187 \\ &0(P1) + 0.038(P_2) - \lambda - \bar{\bar{\mu}}_2 = -2.407 \\ &P_1 + P_2 = 550 \\ &0(P_1) + 1(P_2) = 200 \end{aligned} \quad \text{(E3.29)}$$

In matrix form this set of equations becomes

$$\begin{bmatrix} 0.032 & 0 & -1 & 0 \\ 0 & 0.038 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \\ \bar{\bar{\mu}} \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 550 \\ 200 \end{bmatrix} \quad \text{(E3.30)}$$

The solution is

$$\begin{bmatrix} P_1 \\ P_2 \\ \lambda \\ \bar{\bar{\mu}} \end{bmatrix} = \begin{bmatrix} 350 \\ 200 \\ 13.39 \\ -3.38 \end{bmatrix}$$

Example E 3.6 (The Meaning of μ)

The units of μ are the same as those of λ: $/MWhr. It is important to understand the meaning of

$$\bar{\bar{\mu}}_2 = -3.38\ \$/MWhr$$

This is the incremental cost of the constraint associated with the upper limit of unit 2. It indicates the cost of increasing this limit by 1 MW for the next hour. Since μ_2-double-upper-bar is negative, the "cost" is actually a savings. We can check this conclusion by computing the total system costs when the constraint is 200 MW and when it is 201 MW. For a total demand of 550 MW with a maximum unit-2 generation capability of 200 MW we have P_1 = 350, P_2 = 200 and therefore

$$C_1(P_1) = 0.016(350)^2 + 2.187(350) + 120.312$$
$$C_1(P_1) = 2845.76\ \$/hr$$
$$C_2(P_2) = 0.019(200)^2 + 2.407(200) + 74.074$$
$$C_2(P_2) = 1315.47\ \$/hr$$
$$C_T = C_1(P_1) + C_2(P_2) = 2845.76 + 1315.47 = 4161.23\ \$/hr$$

Now we need to obtain the total costs for a total demand of 550 MW with an increase in the upper limit of the second unit, P_2, by 1 MW. To do this, we need to re-solve the economic dispatch problem. Since we have already found the unconstrained problem (refer to Eq. E3.27) to result in P_1 = 301.71, P_2 = 248.29, we know that the maximum limit of 201 MW limit will be violated. We simply need to adjust Eq. E3.30, resulting in

$$\begin{bmatrix} 0.032 & 0 & -1 & 0 \\ 0 & 0.038 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ \lambda \\ \bar{\bar{\mu}} \end{bmatrix} = \begin{bmatrix} -2.187 \\ -2.407 \\ 550 \\ 201 \end{bmatrix}$$

with a solution of:

$$\begin{bmatrix} P_1 \\ P_2 \\ \lambda \\ \bar{\bar{\mu}} \end{bmatrix} = \begin{bmatrix} 349 \\ 201 \\ 13.35 \\ -3.31 \end{bmatrix}$$

The costs for each generator are, then,

$$C_1(P_1) = 0.016(349)^2 + 2.187(349) + 120.312$$
$$C_1(P_1) = 2832.39\ \$/hr$$
$$C_2(P_2) = 0.019(201)^2 + 2.407(201) + 74.074$$
$$C_2(P_2) = 1325.50\ \$/hr$$
$$C_T = C_1(P_1) + C_2(P_2) = 2832.39 + 1325.50 = 4157.89\ \$/hr$$

The total costs are

$$C_T = C_1(P_1) + C_2(P_2) = 2832.39 + 1325.50 = 4157.89$$

So, as a result of the 1 MW increase in P_{2max}, the total cost will change by

$$4157.89 - 4161.23 = -3.34\ \$/h$$

The small difference between this value and μ-double-upper-bar is due to round-off error and the non-linearity of the problem.

E3.7.2 Graphical Solution

Recall the first KKT condition when applied to the general system eqn.(E3.9) shown again here for convenience:

$$\frac{\partial F}{\partial P_i} = 0 \quad \Rightarrow \quad \frac{\partial C_i(P_i)}{\partial P_i} - \lambda + \underline{\underline{\mu}}_i - \overline{\overline{\mu}}_i = 0 \qquad \forall\, i = 1, n$$

If we assume that all binding inequality constraints have been converted to equality constraint, so that the mu's are zero, then eqn. (E3.32) (above) becomes

$$\frac{\partial F}{\partial P_i} = 0 \quad \Rightarrow \quad \frac{\partial C_i(P_i)}{\partial P_i} - \lambda = 0 \qquad \forall\, i = 1, n \quad i \notin B$$

where B is the set of all generators without binding constraints. This equation implies that for all regulating generators (i.e. units not at their limits) each generator's incremental costs are the same and are equal to λ:

$$\frac{\partial C_1(P_1)}{\partial P_1} = \frac{\partial C_2(P_2)}{\partial P_2} = \ \cdots \ = \frac{\partial C_i(P_i)}{\partial P_i} = \lambda \qquad \forall\, i = 1, n \quad i \notin B$$

This very important principle provides the basis on which to apply the graphical solution method. The graphical solution is illustrated in Figure E3.6 (note that "ICC" means incremental-cost-curve). The unit's data are simply plotted adjacent to each other. Then, a value for λ is chosen (judiciously), a "ruler" is placed horizontally across the graphs at the value of λ, and the generations are added. If the total generation is equal to the total demand "P_T" then the optimal solution has been found. Otherwise, a new value for λ is chosen and the process repeated. The limitations of each unit are included as vertical lines since the solution must not include generation beyond unit capabilities.

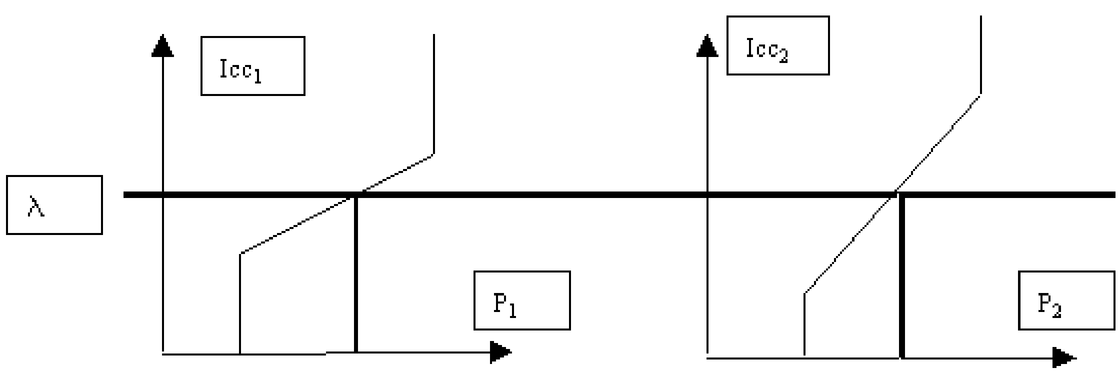

Figure E3.6 Graphical Solution of EDC

Example E 3.7

Our two-unit problem can be solved using a graphical approach as shown in Figure E3.7. The λ axis is on the far right and is used for all units since all units must have the same value for λ at the optimum solution. The ruler yields the generation for each unit at the given value of λ and is shown as a line with solid dots at each end. The ruler can then be used to find the generation for each unit for a given function of λ by moving it up and down. These generation values are then added to find the total generation. If the total generation is the generation to be dispatched, then the placement of the ruler is optimal. Otherwise, the ruler has to be moved up if the total generation is too low, and down if the total generation is too high. To simply the operation, note that the total generation for each value of λ is shown on the far right. Also the λ axis is provided at both the left and right hand sides for convenience. A similar production-costing curve is shown in Figure E3.8 with a ruler which would move in parallel with the above ruler. The solution indicated in Fig E3.7 corresponds to a loading level of about $P_T = 410$ MW, $\lambda = 9.30$ \$/MWhr, $P_1 = 223$ MW, and $P_2 = 187$ Mw. See if you can verify the solutions found in example 3.3 ($P_T = 400$ MW) and 3.5 ($P_T = 550$ MW).

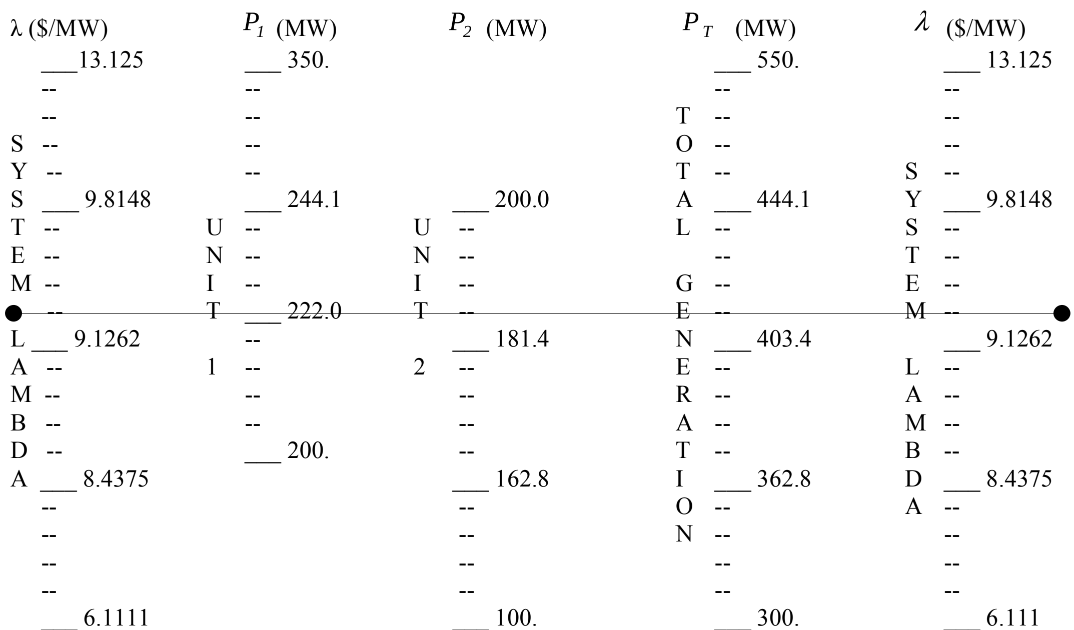

Figure E3.7 Economic Dispatch Graphical Solution

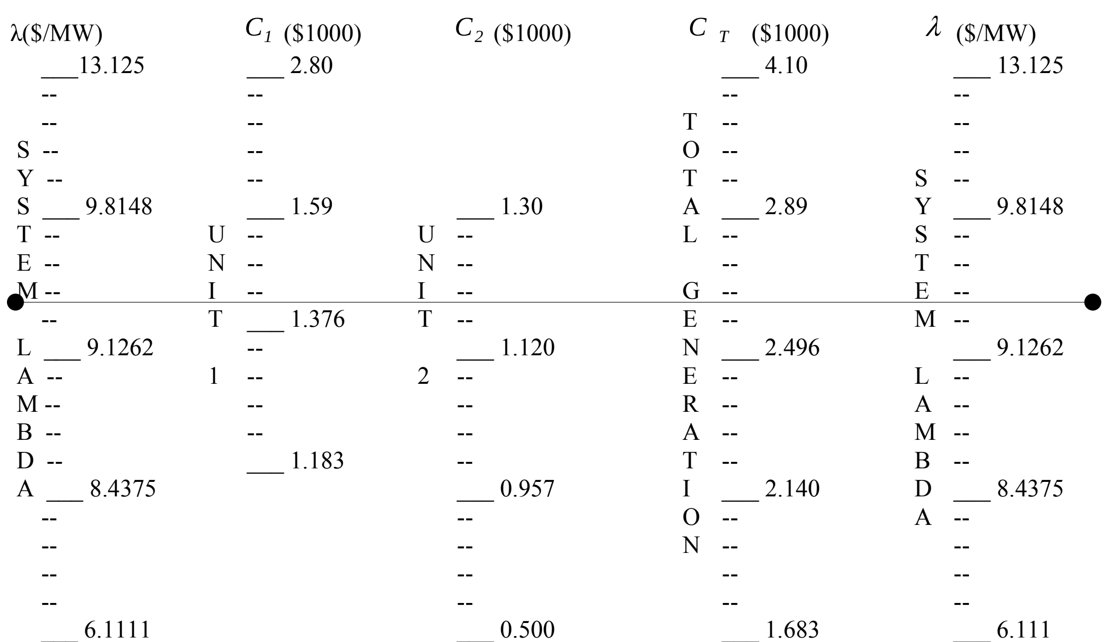

Figure E3.8 Production Costing Graph

E3.8 Summary

The following references are suggested for further reading. This author has used these texts. The texts by Hillier and Lieberman are must reading for all students of Operations Research [3]. This text outlines the application of Linear Programming to many unique problems and even applies special Linear Programming algorithms. The texts by Cooper and Steinberg [1, 2] and by Simmons [6] are texts that are more advanced for Operations Research. Luenberger's text [4] and Pierre's text [5] present the material at a graduate level. Lasdon's text [7] presents advanced material at a graduate level. The classic presentation of optimization for power systems is the text by Wood and Wollenberg [8].

References

1. L. Cooper and D. Steinberg, Introduction to Methods of Optimization, W. B. Saunders Company, Philadelphia, Pennsylvania, 1970.
2. L. Cooper and D. Steinberg, Methods and Applications of Linear Programming, W. B. Saunders Company, Philadelphia, 1974.
3. F. S. Hillier and G. J. Lieberman, Introduction to Operations Research, Holden-Day, Inc., Oakland, California, 1986.
4. D. G. Luenberger, Introduction to Linear and Non-Linear Programming, Addison-Wesley Publishing Company, Reading, Massachusetts, 1973.
5. D. A. Pierre, Optimization Theory with Applications, John Wiley & Sons, Inc., New York, 1969.
6. D. M. Simmons, Nonlinear Programming for Operations Research, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1975.
7. L. S. Lasdon, Optimization Theory for Large Systems, MacMillan Publishing Co., Inc., New York, NY, Collier MacMillan Publishers, London, England, 1970.
8. A. J. Wood and B. F. Wollenberg, Power, Generation, Operation and Control, John Wiley & Sons, New York, NY, 1984.

P R O B L E M S

Problem 1

A two-unit system is given by the following data

$$C_1(P_{g1}) = 0.015 \cdot (P_{g1})^2 + 2 \cdot (P_{g1}) + 6$$
$$C_2(P_{g2}) = 0.025 \cdot (P_{g2})^2 + 7 \cdot (P_{g2}) + 3$$

The total system demand is 500MW. The lower and upper limits for each generator unit are 20 and 300MW, respectively.

(a) Determine the optimal dispatch ignoring inequality constraints
(b) And identify whether it is a feasible dispatch or not (support your answer)

Problem 2

Generator cost rate functions, in $/hr, for a three unit system are given as

$$C_1(P_1) = 0.004P_1^2 + 5.3P_1 + 500$$
$$C_2(P_2) = 0.006P_2^2 + 5.5P_2 + 400$$
$$C_3(P_3) = 0.009P_3^2 + 5.8P_3 + 200$$

Limits on the generation levels are $200 \le P_1 \le 450$, $150 \le P_2 \le 350$, $100 \le P_3 \le 225$. These three generators must supply a total demand of 975 MW.

(a) Form the linear matrix equation necessary to solve the unconstrained optimization problem.
(b) The solution to the unconstrained optimization problem is $P_1 = 482.9MW$, $P_2 = 305.3MW$, $P_3 = 186.5MW$. For this solution (i.e., ignoring limits)
    (i) Compute λ
    (ii) Determine the total cost rate
    (iii) How much would the total cost rate change if the total load increased from 975 to 976 MW? (Indicate whether the total cost rate increases or decreases).
(c) Form the linear matrix equation necessary to solve the next iteration of getting the solution to this problem.

Problem 3

A three-unit system is given by the following data. The total system demand is 1100MW. Generator constraints are $0 < P_{g1} < 550$, $0 < P_{g2} < 300$, $0 < P_{g3} < 300$

$$C_1(P_{g1}) = 0.010 \cdot (P_{g1})^2 + 0.3 \cdot (P_{g1}) + 1$$
$$C_2(P_{g2}) = 0.030 \cdot (P_{g2})^2 + 0.2 \cdot (P_{g2}) + 3$$
$$C_3(P_{g3}) = 0.020 \cdot (P_{g3})^2 + 0.9 \cdot (P_{g3}) + 5$$

(a) Identify the objective function for this optimization problem.
(b) Identify the LaGrangian function assuming no constraints are binding.

(c) Identify the KKT conditions assuming no constraints are binding.
(d) Find the solution to the problem assuming no constraints are binding.
(e) Find the solution to the problem accounting for any binding constraints.
(f) Find the total cost of supplying the 1100MW using the solution found in part (e)
(g) Approximately the total cost of supplying the 1100MW change if the upper limit on generator 1 was increased from 550MW to 560MW.

Problem 4

A three-unit system is given by the following data. The total system demand is 1100MW. Generator constraints are $0 < P_{g1} < 700,\ 0 < P_{g2} < 200,\ 0 < P_{g3} < 252.3$.

$$C_1(P_{g1}) = 0.008 \cdot (P_{g1})^2 + 0.5 \cdot (P_{g1}) + 5$$
$$C_2(P_{g2}) = 0.030 \cdot (P_{g2})^2 + 0.2 \cdot (P_{g2}) + 3$$
$$C_3(P_{g3}) = 0.020 \cdot (P_{g3})^2 + (P_{g3}) + 5$$

(a) Set up the linear matrix equation to solve the economic dispatch problem, assuming all constraints are satisfied (i.e., ignore constraints. DO NOT solve the equation.
(b) The solution to the problem in (a) is $P_{g1} = 664.5MW$, $P_{g2} = 182.2MW$, and $P_{g3} = 253.3MW$. Reformulate this linear matrix equation to solve the economic dispatch problem for this system, accounting for any violated constraints. Again, you DO NOT need to actually solve the equation, just set it up.
(c) Using only the cost function for generator 1, $C_1(P_{g1})$, together with information given in the part b problem statement, determine the system λ for the solution to the unconstrained problem.

Problem 5

Recall that the "system λ" is the cost to the system owner of producing the next MW over the next hour; it is equal to the incremental cost of an individual unit when the system is economically dispatched for minimum cost and the unit is not at an upper or lower generation limit. A two-unit system is given by the following data.

$$C_1(P_{g1}) = 0.015 \cdot (P_{g1})^2 + 2 \cdot (P_{g1}) + 6$$
$$C_2(P_{g2}) = 0.020 \cdot (P_{g2})^2 + 6 \cdot (P_{g1}) + 4$$

The demand is 300MW

1. Write the KKT conditions that must be satisfied at the optimal solution to this problem, assuming that both units are operating between their respective upper and lower limits.
2. Set up the linear matrix equation to solve the economic dispatch problem for this system, assuming that both units are operating between their respective upper and lower limits. Do NOT solve the system of equations.
3. The solution to the problem in (2) is $P_{g1} = 228.57MW$, $P_{g2} = 71.43MW$. Assuming that each unit has a minimum generation capability of 80 MW.
   (a) Indicate why the given solution is not feasible.
   (b) Identify the optimal feasible solution
   (c) Identify the incremental costs of each unit at the optimal feasible solution
   (d) Identify the system λ at the optimal feasible solution
   (e) Would the total cost of supplying the 300MW increase or decrease (relative to the total cost corresponding to the optimal feasible solution) if the minimum generation capabilities on both units were changed to 79MW?

Problem 6

The 'system λ' is the cost to the system owner of producing the next MW over the next hour. It is equal to the incremental cost of an individual unit when the system is economically dispatched for minimum cost and the unit is not at an upper or lower generation limit. A three-unit system is given by the following data.
Total system demand is 1000 MW.

$$C_1(P_{g1}) = 0.008 \cdot (P_{g1})^2 + 0.5 \cdot (P_{g1}) + 5$$
$$C_2(P_{g2}) = 0.015 \cdot (P_{g2})^2 + 2 \cdot (P_{g2}) + 6$$
$$C_3(P_{g3}) = 0.020 \cdot (P_{g3})^2 + P_{g3} + 5$$

a) Set up the linear matrix equation to solve the economic dispatch problem for this system. DO NOT solve the equation.
b) The solution to the problem in (a) is P_{g1} = 549.6 MW, P_{g2} = 243.1 MW, and P_{g3} = 207.3 MW. Assume that each unit has a maximum generation capability of 350 MW. Reformulate the linear matrix equation to solve the economic dispatch problem for this system. Again, DO NOT solve the system.
c) What is the incremental cost for unit 1under the condition specified in part (b)? Do you think the system λ is greater than or less than this value?

Problem 7

Generator 1 has an incremental cost curve of:

$$IC_1(P_{g1}) = 0.05(P_{g1}) + 2.0$$

and limits of:

$$10\ MW \le P_{g1} \le 100\ MW.$$

The generator operates in an economically dispatched system. In this system, it is found that supplying an additional 5 MW costs an additional $50/hr. Determine P_{g1}.

Problem 8

A system consists of two generators supplying a load. Generators 1 and 2 have incremental cost curves as indicated below:

$$IC_1(P_{g1}) = 0.04(P_{g1}) + 2.0$$
$$IC_2(P_{g2}) = 0.06(P_{g2}) + 1.0.$$

and limits of:

$$10\ MW \le P_{g1} \le 100\ MW$$
$$30\ MW \le P_{g2} \le 100\ MW$$

a) In this system, when the load is 140 MW, what is the dispatch of these two units?
b) In this system, when the load is 190 MW, what is the dispatch of these two units?
c) In this system, under a certain economically dispatched scenario (a scenario different than in part (a) and (b)), it is found that supplying an additional 1 MW costs an additional $5.68/hr. Determine P_{g1} and P_{g2}.

Problem 9

A two unit system has incremental cost curves (the derivatives of the cost curves) of $IC_1=0.01P_1+5$, and $IC_2=0.02P_2+4$, where P_1 and P_2 are given in MW. The demand is 300 MW. Ignoring limits on the generators, determine the values of P_1 and P_2 that minimize the cost of supplying the 300 MW.

Problem 10

A two-generator system is operating on economic dispatch and supplying 420 Mw of load. The total cost of supply is computed from the final EDC solution (i.e., all constraints are satisfied) and found to be \$3000/hr. From this same final solution, the LaGrange multipliers are found to be:

Equality constraint	λ=\$15/Mw-hr
$Pg_1 \geq 20$ Mw	$\mu_{1,L}=0$
$Pg_1 \leq 300$ Mw	$\mu_{1,H}=0$
$Pg_2 \geq 10$ Mw	$\mu_{2,L}=0$
$Pg_2 \leq 200$ Mw	$\mu_{2,H}=$ -\$4.00/Mw-hr

Here the subscripts "L" and "H" indicate "Low limit" and "High limit," respectively, and refer to the corresponding inequality constraint. For each question below, you must provide some basis or reasoning for your response.

(a) What would be the (approximate) total cost of supply if the total demand was increased to 421 MW?
(b) What would be the total cost of supply if the lower limit for generator 2 was increased from 10 MW to 11 MW?
(c) What would be the total cost of supply if the upper limit for generator 2 was increased from 200 MW to 201 MW?
(d) What are the generation levels in Mw of generators 1 and 2 ?
(e) What is the incremental cost for generator 1?

Problem 11

A two generator system has cost curves (\$/hr) of $C_1(P_1)=0.006P_1^2+5P_1+3$, and $C_2(P_2)=0.01P_2^2+4P_2+2$, where P_1 and P_2 are given in MW. The total demand is $P_T=500$ MW. The limits on these generators are $0\leq P_1\leq 300$ and $0\leq P_2\leq 300$.

a. Determine the unconstrained values of P_1 and P_2 that minimize the cost of supplying the 500 MW, and indicate whether this solution is feasible or not.
b. For the solution found in (a), how much would the total cost of supply change if the total demand increased to 501 MW for one hour ?
c. Use the complementary condition (the third condition in the KKT conditions), to identify the values of each Lagrange multiplier associated with the inequality constraints.